CfPA-97-TH-
IC-96/97-39

The closet non-Gaussianity of anisotropic Gaussian fluctuations

Pedro G. Ferreira[1] and João Magueijo[2]
(1) *Center for Particle Astrophysics, University of California, Berkeley CA 94720-7304,USA*
(2) *The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK*

In this paper we explore the connection between anisotropic Gaussian fluctuations and isotropic non-Gaussian fluctuations. We first set up a large angle framework for characterizing non-Gaussian fluctuations: large angle non-Gaussian spectra. We then consider anisotropic Gaussian fluctuations in two different situations. Firstly we look at anisotropic space-times and propose a prescription for superimposed Gaussian fluctuations; we argue against accidental symmetry in the fluctuations and that therefore the fluctuations should be anisotropic. We show how these fluctuations display previously known non-Gaussian effects both in the angular power spectrum and in non-Gaussian spectra. Secondly we consider the anisotropic Grischuk-Zel'dovich effect. We construct a flat space time with anisotropic, non-trivial topology and show how Gaussian fluctuations in such a space-time look non-Gaussian. In particular we show how non-Gaussian spectra may probe superhorizon anisotropy.

PACS Numbers : 98.80.Cq, 98.70.Vc, 98.80.Hw

I. INTRODUCTION

Anisotropic models of the Universe have been often considered in the past (eg. [1,2]). In recent times globally anisotropic spacetimes have attracted attention for their thought provoking value, as primordial anisotropy would appear to contradict inflation [3]. It is therefore important to find experimental evidence for, or constraints on, primordial anisotropy. The cosmic microwave background (CMB) is the cleanest and most accurate experimental probe in current cosmology. Thus it makes sense to explore the impact of anisotropic expansion on the CMB. For homogeneous space times this was largely done in [4–6]. In the more sophisticated analysis in [6] the effects of the unperturbed anisotropic expansion were combined with a spectrum of superposed Gaussian fluctuations. An admitted shortcoming of this analysis is the assumption that while the unperturbed model leaves an anisotropic pattern in the sky, the Gaussian fluctuations around it are isotropic. Should the Gaussian fluctuations in such models be anisotropic one may expect a more stringent statistical bound on anisotropy, if the Universe is indeed isotropic. One can consider another class of models where the background space-time is homogeneous and isotropic but anisotropic topological identifications lead to anisotropic Gaussian fluctuations. Some of these universes have been considered before [7] and an example of the patterns in an open universe has been presented in [8].

The apparently unrelated issue of large-angle CMB non-Gaussianity has also been considered recently, both as an experimental matter [9], and as a possible prediction in topological defect theories [10–16]. In [10], in particular, an outline is given of a comprehensive formalism for encoding large angle non-Gaussianity based on the spherical harmonic coefficients a^ℓ_m in the expansion

$$\frac{\Delta T(\mathbf{n})}{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a^\ell_m Y^\ell_m(\mathbf{n}) \tag{1}$$

In [10] it is also stated that "any non-Gaussian theory is to some extent anisotropic, favouring particular directions in the sky and some m's over others". The converse statement follows: that Gaussian anisotropic fluctuations will appear as non-Gaussian fluctuations from the standpoint of an isotropic theory. This establishes an interesting link between the search for cosmological anisotropy and the search for non-Gaussian signatures.

Let us consider Gaussian theories which favour an axis, where Ω are angles defining this axis. Then the probability distribution conditional to this axis $P(a^\ell_m|\Omega)$ is Gaussian. Isotropy is violated, but the resulting theory is Gaussian within the reduced set of symmetries the theory now must satisfy. However from an isotropic point of view the full ensemble is made up of all the ensembles which favour an axis, but allowing the axis to be uniformly distributed. Such a super-ensemble would undoubtedly be isotropic, but it would also be non-Gaussian. Marginalizing with respect to the axis reveals a non-Gaussian theory, that is

$$\begin{aligned} P(a^\ell_m) &= \int d\Omega P(\Omega) P(a^\ell_m|z_1) \\ P(\Omega) &= \frac{\sin\theta}{4\pi} \end{aligned} \tag{2}$$

is non-Gaussian. This identifies the origin of the Gaussian/non-Gaussian switch. Conditionalizing to an axis renders the theory Gaussian (and anisotropic). Marginalizing with respect to the axis reveals a non-Gaussian theory (but an isotropic ensemble).

This phenomenon turns out to be a particular case of the general phenomenon discussed in connection with the texture analytical model in [11,12]. In that model it is found that the temperature anisotropies are very non-Gaussian. The theory has C_ℓ cosmic variance error bars above their Gaussian value, and there are strong correlations among C_ℓ. It turns out, however, that these large-angle non-Gaussian effects are largely due to the last texture (as in the texture closest to us, or the texture at lower redshift). The culprit identified, one then notices that conditionalizing the theory to the last texture redshift z_1 reveals a Gaussian ensemble, that is, the probability distribution $P(a^\ell_m|z_1)$ is Gaussian. Marginalizing with respect to z_1, however, produces a non-Gaussian ensemble, that is the probability

$$P(a^\ell_m) = \int dz_1 P(z_1) P(a^\ell_m|z_1) \tag{3}$$

is non-Gaussian.

The picture is then clear [17]. We come up with a construction where the full ensemble is made up of sub-ensembles which are Gaussian. Each sub-ensemble is however labelled by an index which from the the point of view of the full ensemble is a random variable. Marginalizing with respect to this variable reveals a non-Gaussian ensemble. Conditionalizing with respect to this index renders the theory Gaussian. Such an index was called in [12] the random index, and it was conjectured [1] in that paper that non-Gaussianity could often be characterized by a set of such indices labelling Gaussian ensembles. Within such a construction the strategy for predicting experiment must be modified. One should now not provide a direct statistical description of the full ensemble (that is, marginal distributions), which would be plagued by all sorts of non-Gaussian effects. Rather it makes more sense to supply information on all the Gaussian sub-ensembles, plus the distribution function of their random indices.

Hence we may use a sub-class of the comprehensive formalism for encoding large-angle non-Gaussianity outlined in [10] to describe anisotropic Gaussian fluctuations. This is essentially a large-angle generalization of [19] and is described in Section II. The idea is to complement the angular power spectrum C_ℓ with a set of multipole shape spectra B_m describing how the power is distributed among the m's for a given scale l. The B_m encode information on the shape of large angle structures. They are uniformly distributed in a Gaussian isotropic theory, meaning its fluctuations are shapeless. However, as we shall see in Section V, preferred shapes emerge in non-Gaussian isotropic theories, as well as in Gaussian anisotropic theories, where the B_m are not uniformly distributed. Non-Gaussian spectra then appear as a natural predictive tool for these theories.

In this paper we study the disguised non-Gaussianity of anisotropic Gaussian fluctuations along two lines. Firstly, in Section III, we propose a simple method for defining anisotropic Gaussian fluctuations. Breaking isotropy essentially amounts to choosing an alternative symmetry group under which the covariance matrix should be invariant, and which picks a favoured direction in the sky. We can then write down the most general form for the covariance matrix of the theory simply by studying the representation theory of the symmetry group. We argue that the accidental symmetry allowing anisotropic fluctuations to be isotropic is a model dependent and unnatural assumption. Hence Gaussian fluctuations in anisotropic Universes should be anisotropic too. Although we concentrate on anisotropic fluctuations with an $SO(2)$ symmetry, the definition and considerations given in Section II are quite general, as explained in more detail in Appendix I.

We then show how anisotropic Gaussian theories induce well known non-Gaussian effects in the relation between the observed and the predicted angular power spectrum C_ℓ. These effects include larger cosmic variance error bars, and also the phenomenon of cosmic covariance, that is correlations between the observed C_ℓ. Cosmic covariance allows for more structure to exist in each realization than in the predicted average power spectrum and complicates comparison between theory and experiment. These effects are shown to be present for anisotropic Gaussian theories in Section IV.

Then, in Section V, we show how anisotropic Gaussian fluctuations render non-Gaussian spectra non-uniformly distributed, as announced above. We also find the most general class of isotropic non-Gaussian theories into which anisotropic Gaussian fluctuations may be mapped. As a concrete example in Section VI we proceed to characterize the non-Gaussian spectra for the relevant, globally anisotropic space times.

Along a totally different line in Section VII we construct a simple example of a topologically non-trivial space time and show how the non-Gaussian spectra will indicate anisotropic topological identifications. We propose this as an

[1] This conjecture can in fact be promoted to a mathematical theorem; see [18].

anisotropic Grischuk-Zel'Dovich effect: from subhorizon, large angle observables we can characterize super-horizon anisotropies.

In Section VIII we discuss the implications of our results and their practical implementation.

II. LARGE-ANGLE NON-GAUSSIANITY

We now set up a formalism for describing large-angle non-Gaussianity which is based on [19], but makes use of a^ℓ_m coefficients rather than Fourier components, and so is suitable for mapping large-angle non-Gaussianity. Again the idea is to map the $\{a^\ell_m\}$ into a set of spectra which for a Gaussian isotropic theory are independent random variables. One of these spectra is the angular power spectrum C_ℓ, and should be a $\chi^2_{2\ell+1}$ for a Gaussian isotropic theory. The other variables make up non-Gaussian spectra which should be uniformly distributed for a Gaussian isotropic theory.

The transformation proposed is defined as follows. Firstly we split the complex modes into moduli and phases

$$
\begin{aligned}
a^\ell_0 &= s^l_0 \rho^\ell_0 \\
a^\ell_m &= \frac{\rho^\ell_m}{\sqrt{2}} e^{i\phi^\ell_m}
\end{aligned}
\tag{4}
$$

where $s^l_0 = \pm 1$ is simply the sign of a^ℓ_0. The fact that the $m = 0$ mode is real introduces a slight modification to the construction in [19]. There are now $\ell + 1$ moduli, but there are only ℓ phases (the index m starts at 1 for the phases). Working out the Jacobian of the transformation shows that for a Gaussian theory the distribution of the $\{\rho^\ell_m, \phi^\ell_m, s^\ell_0\}$ is

$$
F(\rho^\ell_m, \phi^\ell_m, s^\ell_0) = \frac{2\exp\left(-\frac{\sum_0^\ell \rho^2_m}{2C_\ell}\right)}{(2\pi)^{1/2} C_\ell^{\ell+1/2}} \times \left(\prod_1^\ell \rho_m\right) \times \frac{1}{(2\pi)^\ell} \times \frac{1}{2} \tag{5}
$$

The phases ϕ^ℓ_m are uniformly distributed in $[0, 2\pi]$. The sign s^ℓ_0 has a uniform discrete distribution. The moduli ρ^ℓ_m are χ^2_2 distributed except for ρ^ℓ_0 which is χ^2_1 distributed. Since ρ_0 now does not appear in the Jacobian of the transformation, the only way one can proceed with the construction in [19] is by ordering the ρ's by decreasing order of m, and then introduce polars:

$$
\begin{aligned}
\rho^\ell_\ell &= r\cos\theta_1 \\
\rho^\ell_{\ell-1} &= r\sin\theta_1\cos\theta_2 \\
&\dots \\
\rho^\ell_1 &= r\sin\theta_1 \dots \cos\theta_l \\
\rho^\ell_0 &= r\sin\theta_1 \dots \sin\theta_l
\end{aligned}
\tag{6}
$$

Again, working out the jacobian of the transformation implies that for a Gaussian isotropic theory the distribution of these variables is

$$
F(r, \theta_m, s^\ell_0, \phi_m) = \frac{\exp\left(-r^2/2C_\ell\right) r^{2\ell}}{(\pi/2)^{1/2} C_\ell^{\ell+1/2}} \prod_1^\ell \cos\theta_m (\sin\theta_m)^{2(\ell-i)} \times \frac{1}{2} \times \frac{1}{(2\pi)^\ell} \tag{7}
$$

One can then define shape spectra B^ℓ_m as

$$
B^\ell_m = (\sin\theta_m)^{2(\ell-m)+1} \tag{8}
$$

so that for a Gaussian isotropic theory one has:

$$
F(r, B^\ell_m, s^\ell_0, \phi^\ell_m) = \frac{\exp\left(-r^2/2C_\ell\right) r^{2\ell}}{(\pi/2)^{1/2} C_\ell^{\ell+1/2} (2\ell-1)!!} \times \frac{1}{2} \times \frac{1}{(2\pi)^\ell} \tag{9}
$$

The angular power spectrum C^ℓ seen as a random variable is then related to r by

$$
C^\ell = \frac{r^2}{2\ell+1} \tag{10}
$$

and is a $\chi^2_{2\ell+1}$. The multipole shape spectra B^ℓ_m may be obtained from the moduli ρ^ℓ_m according to

$$B^\ell_m = \left(\frac{\rho^{\ell 2}_{m-1} + \cdots + \rho^{\ell 2}_0}{\rho^{\ell 2}_m + \cdots + \rho^{\ell 2}_0}\right)^{\ell - m + 1/2} \tag{11}$$

and are uniformly distributed in $[0, 1]$. Finally the phases ϕ^ℓ_m are uniformly distributed in $[0, 2\pi]$, and the sign s^ℓ_0 is a discrete uniform distribution over $\{-1, +1\}$.

As in [19] we define non-Gaussian structure in terms of departures from uniformity and independence in the $\{B^\ell_m, \phi^\ell_m\}$. Gaussian theories can only allow for modulation, that is, a non-constant power spectrum. The most general power spectrum has as much information as Gaussian theories can carry. White-noise is the only type of fluctuations which is more limited in terms of structure than Gaussian fluctuations [2]. In isotropic Gaussian theories there is no structure in the $\{B^\ell_m, \phi^\ell_m\}$ since these are independent and uniformly distributed. By allowing the B^ℓ_m to be not uniformly distributed, or to be constrained by correlations amongst themselves and with the power spectrum, one adds shape to the multipoles. This is because the B^ℓ_m tell us how the power in multipole ℓ given by C^ℓ (or r) is distributed among the various $|m|$ modes, which reflect the shape of the fluctuations. Indeed the $m = 0$ mode (zonal mode) has no azimuthal dependence. It corresponds to fluctuations with strict cylindrical symmetry (rather than statistical symmetry). The $|m| > 0$ modes correspond to the various azimuthal frequencies allowed for the scale ℓ. Each of these modes represent a way in which strict cylindric symmetry may be broken. The relative intensities of all the m modes carry information on the shape of the random structures at least as seen by the scale ℓ. In a Gaussian theory all the m modes must have the same intensity, something which can be rephrased by the statement that the B^ℓ_m are independent and uniformly distributed. Hence Gaussian fluctuation display shapeless multipoles. Any departure from this distribution in the B^ℓ_m may then be regarded as a evidence for more or less random shape in the fluctuations.

On the other hand the phases ϕ^ℓ_m transform under azimuthal rotations. Therefore they carry information on the localization of the fluctuations. If the phases are independent and uniformly distributed then the perturbations are delocalized.

Finally there may be correlations between the various scales defined by ℓ. In the language of [19] this is what is called connectivity of the fluctuations. These correlations measure how much coherent interference is allowed between different scales, a phenomenon required for the rather abstract shapes and localization on each scale to become something visually recognizable as shapeful or localized. As in [19] this may be cast into inter-ℓ correlators. As we shall see these are in fact quite complicated for general anisotropic Gaussian theories. Therefore we have chosen not to dwell on this aspect of large-scale non-Gaussianity in this paper.

III. A POSSIBLE METHOD FOR INTRODUCING GAUSSIAN FLUCTUATIONS IN ANISOTROPIC UNIVERSES

We now present a possible way of introducing Gaussian fluctuations in anisotropic Universes such as the Bianchi models. In Section VII we will present another context in which anisotropy appears: periodic Universes. There we shall present more specific calculations of anisotropic Gaussian perturbations. Here we shall however use a method which relies simply on inspecting the reduced symmetry group anisotropic Gaussian perturbations must satisfy. This is a simple, if somewhat phenomenological, way of introducing the most general Gaussian perturbation which can live in an anisotropic background. Without actually performing a detailed perturbation analysis of these spacetimes, one can refine the analysis of [6] by using this prescription and possibly find more stringent constraints.

Let an all-sky temperature anisotropy map be decomposed into spherical harmonics as in Eqn. (1). Then, for a general Gaussian theory, the a^ℓ_m are Gaussian random variables specified by a covariance matrix which must satisfy the symmetries of the underlying theory. In Friedman models the symmetry group is $SO(3)$, but the symmetry group may be smaller. Anisotropic Gaussian fluctuations may be defined as Gaussian fluctuations with a covariance matrix satisfying a symmetry group which picks a favoured direction in the sky. We concentrate on anisotropic fluctuations with an $SO(2)$ symmetry, that is, with cylindrical symmetry.

[2] It is curious to note that white noise has less structure than generic Gaussian fluctuations, but it also has more symmetry. It is tempting to associate reduction of symmetry and addition of structure. Anisotropic fluctuations have less symmetry than isotropic fluctuations, but they also have more structure, reflected in their non-Gaussian structure.

The general form of the covariance matrix may be obtained just from the representation theory of the symmetry group. The symmetry group breaks the $\{a^\ell_m\}$ space into irreducible representations (Irreps). The a^ℓ_m may then be reexpressed in a basis adapted to these Irreps. Using Schur's Lemmas [20] one knows (see Appendix I for more detail) that the covariance matrix of the theory must be a multiple of the identity within each Irrep [3]. Furthermore correlations between different a^ℓ_m can only occur for elements of different but equivalent Irreps. Hence, for any Gaussian theory subject to a symmetry which does not lead to equivalent Irreps, the spherical harmonic coefficients, expressed in a basis adapted to the partition into Irreps, must be independent random variables, and their variance must be a function only of the Irrep they belong to. As we shall see it may happen that the variance is the same for a set of Irreps. This degeneracy then leads to an accidental enlarged symmetry. If some of the Irreps are equivalent then in principle one may also have correlations between coefficients belonging to different but equivalent Irreps.

As an example consider an isotropic theory. Then the $\{a^\ell_m\}$ for each ℓ are an Irrep of the symmetry group $SO(3)$ represented by the D matrices

$$R(\psi,\theta,\phi)a^\ell_m = D^\ell_{mm'}(\psi,\theta,\phi)a^\ell_{m'} \tag{12}$$

where (ψ,θ,ϕ) are Euler angles. None of these Irreps is equivalent, as indeed none of them have the same dimension. Hence for a Gaussian isotropic theory the a^ℓ_m must have a covariance matrix of the form

$$\langle a^\ell_m a^{\ell'\star}_{m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell \tag{13}$$

If the angular power spectrum C_ℓ happens to be a constant (white-noise) over a certain section of the spectrum then this degeneracy increases the symmetry group of the theory: rotations among different ℓ's are now an extra symmetry. This is an accidental symmetry resulting from the degeneracy displayed by the particular model considered (white noise) and not required by the underlying theory.

Now suppose that the symmetry group is $SO(2)$, that is, the unperturbed model supporting the fluctuations is cylindrically symmetric. Then there is a favoured axis in the Universe and with respect to this axis the symmetry transformations are

$$R(\phi)a^\ell_m = e^{im\phi}a^\ell_m \tag{14}$$

The Irreps are now indexed by ℓ, m with $m \geq 0$. They are one dimensional complex Irreps for $m > 0$, and one dimensional real (and trivial) Irreps for $m = 0$. For the same m Irreps with different ℓ are equivalent Irreps. For each ℓ we have a single Irrep of $SO(3)$ which splits into $\ell + 1$ Irreps of $SO(2)$. The covariance matrix of the theory now has the general form:

$$\langle a^\ell_m a^{\ell'\star}_{m'}\rangle = \delta_{mm'}C^{\ell\ell'}_{|m|} \tag{15}$$

and we may call the diagonal terms $C_{\ell m}$ of $C^{\ell\ell'}_{|m|}$ the cylindrical power spectrum. It may now happen that $C^{\ell\ell'}_{|m|} = \delta^{\ell\ell'}C^\ell_{|m|}$, and furthermore that a given model displays the degeneracy $C_{\ell|m|} = C_\ell$, that is the cylindrical power spectrum is white noise in m. In this case the $SO(3)$ symmetry is accidentally restored. However this is no different from the white-noise model $C_\ell =$ const referred to above. It is merely an accidental enlarged symmetry displayed by a concrete model and not a fundamental symmetry imposed by the underlying model.

Accidental symmetries (eg. family symmetry in particle physics) are always regarded with horror. If they happen to exist, sooner or later a fundamental principle is sought which will promote them from accidental to fundamental symmetries. If they don't happen to exist a priori, such as in the case of fluctuations in anisotropic models, then better not postulate them in the first place.

IV. NON-GAUSSIAN EFFECTS ON THE ANGULAR POWER SPECTRUM

Gaussian anisotropic theories display many of the novelties present in non-Gaussian theories, such as the texture models considered in [11,12]. They trade their added predictivity in terms of non-Gaussian spectra for larger cosmic

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[3] Schurs' Lemma only applies to finite dimensional representations, such as the ones offered by the a^ℓ_m. If one instead looks at the real space maps $\delta T/T$, then the representation space is S^2. This is infinite dimensional, and indeed the covariance matrix of Gaussian theories is not diagonal, and is specified by the two-point correlation function $C(\theta)$.

variance error bars in the angular power spectrum. Also the observed C_ℓ may be correlated, a phenomenon called cosmic covariance and present in the texture models in [11,12]. Cosmic covariance (or C^ℓ aliasing) induces great mess when comparing predicted and observed power spectra. Correlations allow for each observed power spectrum to have more structure than the average power spectrum. This may result in the average power spectrum corresponding to nothing that any observer ever sees. More subtle methods for predicting power spectra are then necessary. Two prescriptions are given in [12].

A. Cosmic variance surplus

For a Gaussian isotropic theory the angular power spectrum

$$C^\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |a^\ell_m|^2 \tag{16}$$

has the variance

$$\sigma^2(C^\ell) = \frac{2C_\ell^2}{2\ell+1} \tag{17}$$

Here we use the notation C^ℓ to denote the random variable and C_ℓ to denote its ensemble average. For a Gaussian anisotropic theory this variance is

$$\sigma^2(C^\ell) = \frac{2}{(2\ell+1)^2}\sum_{m=-\ell}^{\ell} C^2_{\ell m} \tag{18}$$

If we define the average cylindrical power spectrum by

$$C_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} C_{\ell m} \tag{19}$$

then

$$\sigma^2(C^\ell) \geq \frac{2C_\ell^2}{2\ell+1} \tag{20}$$

It is a simple analysis exercise to prove this inequality and show that it is saturated only when $C_{\ell m} = C_\ell$, that is when the fluctuations are isotropic.

Generally we may interpret this result as a reduction in the number of degrees of freedom in the χ^2 induced by anisotropy. Suppose, for instance, that a theory is strongly anisotropic so that only a few m modes among the available $2\ell+1$ contribute to the power spectrum C_ℓ, for a given ℓ. Then, effectively, the observed power spectrum C^ℓ is the result of these few modes. Since these are still Gaussian variables the observed power spectrum is a χ^2, but with an effective number of degrees of freedom equal to the number of predominant modes. If for example all the power is concentrate on the $m=0$ mode, then the C^ℓ is a χ^2_1. If all the power is in a $m>0$ mode, the C^ℓ is a χ^2_2.

We may use the ratio between the actual cosmic variance of the theory and its Gaussian prediction to quantify how anisotropic the fluctuations are. Quantitatively let us call anisotropy in the multipole ℓ to the quantity

$$A_\ell = \frac{\sigma^2_{\mathrm{GA}}(C^\ell)}{\sigma^2_{\mathrm{GI}}(C^\ell)} = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}\left(\frac{C_{\ell m}}{C_\ell}\right)^2 \tag{21}$$

which varies between $A_\ell = 1$ for isotropic theories to $A_\ell = 2\ell+1$ for cylindrically symmetric multipoles (for which all the power is in the $m=0$ mode).

B. Cosmic covariance

There are also correlations between different C^ℓ. For $\ell \neq \ell'$ we have that

$$\mathrm{cov}(C^\ell, C^{\ell'}) = \frac{1}{(2\ell+1)(2\ell'+1)} \sum_{m,m'} \mathrm{cov}(|a^\ell_m|^2, |a^{\ell'}_{m'}|^2) \tag{22}$$

For two (possibly correlated) complex Gaussian random variables z_1 and z_2 with uncorrelated real and imaginary parts, it can be shown that $\mathrm{cov}[|z_1|^2, |z_2|^2] = \langle z_1 z_2^* \rangle^2 + \langle z_1 z_2 \rangle^2$, and so

$$\mathrm{cov}(C^\ell, C^{\ell'}) = \frac{1}{(2\ell+1)(2\ell'+1)} \sum_m C^{\ell\ell' 2}_m \tag{23}$$

where m in the summation runs from $-\min(\ell, \ell')$ to $\min(\ell, \ell')$. The off-diagonal elements (in ℓ, ℓ') in $C^{\ell\ell'}_m$ therefore induce correlations among the various observed C^ℓ. A possible, but model dependent, way to do away with these correlations is to rotate the C^ℓ among themselves so as to diagonalize the covariance matrix (23). These rotated C^ℓ will then be independent, and so their average value is a good prediction for what each observer will see. Also, as shown in [12], in the rotated basis the cosmic variance error bars tend to be smaller and approach their Gaussian minimum. Therefore cosmic covariance, and larger cosmic variance error bars can be dealt with by means of this trick. However this trick does depend on each particular model, and is not a Universal prescription applicable to every model.

V. THE NON-GAUSSIAN STRUCTURES EXHIBITED BY ANISOTROPIC GAUSSIAN THEORIES

Anisotropic Gaussian theories also display non-Gaussian structure in the senses given at the end of Sec.II, that is they produce non-trivial non-Gaussian spectra. Here we shall find the most general type of isotropic non-Gaussian structure which can be mapped from these theories.

We shall consider the anisotropic covariance matrix in more detail. Let the matrix $C^{\ell\ell'}_m$ be split into its diagonal and its off-diagonal $X^{\ell\ell'}_m$ parts

$$C^{\ell\ell'}_m = \delta^{\ell\ell'} C_{\ell|m|} + X^{\ell\ell'}_m \tag{24}$$

Then $X^{\ell\ell'}_m \ll C_{\ell|m|}$, and so the bilinear form in the exponent of the Gaussian distribution

$$F(a^\ell_m) \propto \exp\left(-\sum_m \sum_{\ell,\ell'} a^\ell_m M^{\ell\ell'}_m a^{\ell'}_m \right) \tag{25}$$

is

$$M^{\ell\ell'}_m = C^{\ell\ell' -1}_m = \frac{\delta^{\ell\ell'}}{C_{\ell|m|}} - \frac{X^{\ell\ell'}_m}{C_\ell C_{\ell'}} \tag{26}$$

and so the distribution factorizes into a factor which reveals the structure inside each multipole, and a factor which reveals correlations between different multipoles. We shall analyze these two factors in turn.

Let's first assume that $X^{\ell\ell'}_m = 0$. Repeating the transformation presented in Section II but using a covariance matrix of the form (15) one ends up with a rather complex distribution which has the form:

$$F(C^\ell, B^\ell_m, s^\ell_0, \phi^\ell_m) = F(C^\ell, B^\ell_m) \times \frac{1}{2} \times \frac{1}{(2\pi)^\ell} \tag{27}$$

Unless $C_{\ell m} = C_\ell$, the B^ℓ_m are not uniformly distributed. Also the C^ℓ will in general not be a $\chi^2_{2\ell+1}$, and the function $F(C^\ell, B^\ell_m)$ will not factorize. This means that not only will correlations exist between the B^ℓ_m but the B^ℓ_m will also be correlated with the angular power spectrum. The phases ϕ^ℓ_m on the other hand will still be uniformly distributed and independent. The phases tell us nothing about Gaussian anisotropic fluctuations.

Hence anisotropic Gaussian fluctuations, when seen from the point of view of an isotropic formalism, are an example of delocalized shapeful fluctuations (explored in some detail in [19]). In the next two sections we will explore in more

detail the particular type of non-Gaussian effects which Gaussian anisotropic fluctuations may induce. The shapes exhibited by these theories are not the most general shapes, because there must be a scale transformation in the ρ^ℓ_m which would render the B^ℓ_m uniformly distributed again. Clearly not all shapes have this property.

On top of this if $X^{\ell\ell'}_m \neq 0$ the distribution $F(a^\ell_m)$ does not factorize into factors which only depend on one ℓ. Correlations between the different ℓ will then appear, which in the language of [19] amount to the emergence of connected structures: different scales are allowed to interfere constructively. In this paper we will not explore this side of the problem in depth. Nevertheless we have identified the non-Gaussian structures into which anisotropic Gaussian fluctuations are mapped. These are the delocalized shapeful (and possibly connected) structures defined in [19], or rather, a subclass thereof.

We should note that although the $\{C^\ell, B^\ell_m, \phi^\ell_m\}$ decomposition is not $SO(3)$ invariant, the $\{C^\ell, B^\ell_m\}$ already are $SO(2)$ invariant [4]. Since the phases contain no information whatsoever on Gaussian anisotropic fluctuations they do not count as a device for making predictions in these theories (as much as one does not compute B^ℓ_m for Gaussian isotropic theories). Hence the set of variables $\{C^\ell, B^\ell_m\}$ is suitable for representing invariantly the most general form of non-Gaussian fluctuation which can be mapped from Gaussian anisotropic fluctuations.

VI. GLOBALLY ANISOTROPIC UNIVERSES

A useful set of models in which to explore these concepts are the homogeneous, anisotropic cosmologies, also know as the Bianchi models [1]. One can describe Bianchi cosmologies in terms of the metric

$$g_{\mu\nu} = -n_\mu n_\nu + a^2[\exp(2\beta)]_{AB} e^A_\mu e^B_\nu, \tag{28}$$

where n_α is the normal to spatial hypersurfaces of homogeneity, a is the conformal scale factor, β_{AB} is a 3 matrix only dependent on cosmic time, t, and e^A_μ are invariant covector fields on the surfaces of homogeneity, which obey the commutation relations

$$e^A_{\mu;\nu} - e^A_{\nu;\mu} = C^A_{BC} e^B_\mu e^C_\nu. \tag{29}$$

The structure constants C^A_{BC} can be used to classify the different models. We shall focus on open or flat models which are asymptotically Friedman. These can be obtained by taking different limits of the type VII$_h$ model which has structure constants

$$C^2_{31} = C^3_{21} = 1,\ C^2_{21} = C^3_{31} = \sqrt{h} \tag{30}$$

It is convenient to define the parameter $x = \sqrt{h/(1-\Omega_0)}$, which determines the scale on which the principal axes of shear and rotation change orientation. By taking combinations of limits of Ω and x one can obtain Bianchi I, V and VII$_0$ cosmologies.

We are interested in large-scale anisotropies so it suffices to evaluate the peculiar redshift a photon will feel from the epoch of last scattering (ls) until now (0)

$$\Delta T_A(\hat{\mathbf{r}}) = (\hat{r}^i u_i)_0 - (\hat{r}^i u_i)_{ls} - \int_{ls}^0 \hat{r}^j \hat{r}^k \sigma_{jk} d\tau \tag{31}$$

where $\hat{\mathbf{r}} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ is the direction vector of the incoming null geodesic, $\mathbf{u}$ is the spatial part of the fluid four-velocity vector and to first order, the shear is $\sigma_{ij} = \partial_\tau \beta_{ij}$. To evaluate expression (31), one must first of all determine a parameterization of geodesics on this spacetime. This is given by

$$\begin{aligned}
\tan(\frac{\phi(\tau)}{2}) &= \tan(\frac{\phi_0}{2})\exp[-(\tau-\tau_0)\sqrt{h}] \\
\theta(\tau) &= \theta_0 + (\tau-\tau_0) \\
-\frac{1}{\sqrt{h}}\ln\{\sin^2(\frac{\phi_0}{2}) &+ \cos^2(\frac{\phi_0}{2})\exp[2(\tau-\tau_0)\sqrt{h}]\}
\end{aligned} \tag{32}$$

[4]We are assuming that not only the Universe is anisotropic but that we know, a priori, what its symmetry axis is, eg: by the detection of a Hubble-size coherent magnetic field. Alternatively we leave the Euler angles of this axis free, to be estimated by some MLE.

Solving Einstein's equations (and assuming that matter is a pressureless fluid) one can determine $\mathbf{u}$ and σ_{ij}. A general expression for (31) was determined in [5]:

$$\Delta T_A(\hat{\mathbf{r}}) = (\frac{\sigma}{H})_0 \frac{2\sqrt{1-\Omega_o}}{\Omega_0} \times \{[\sin\phi_0\cos\theta_0 - \sin\phi_{ls}\cos\theta_{ls}(1+z_{ls})]$$
$$- \int_{\tau_{ls}}^{\tau_0} \frac{3h(1-\Omega_0)}{\Omega_0} \sin 2\phi[\cos(\theta)+\sin(\theta)] \frac{d\tau}{\sinh^4(\sqrt{h}\tau/2)}\} \tag{33}$$

A useful discussion of the different CMB patterns imprinted by the unperturbed anisotropic expansion is presented in [5]. The patterns can be roughly said to be constructed out of two ingredients: a focusing of the quadrupole when $\Omega < 1$ and a spiral pattern when x is finite. The Bianchi VII$_h$ is most general form of homogeneous, anisotropic universes in an $\Omega \leq 1$ which are asymptotically Friedman-Robertson-Walker. The pattern is of the form:

$$\frac{\Delta T}{T} = f_1(\theta)\cos(\phi - \tilde{\phi}(\theta)) \tag{34}$$

In each $\theta =$ const circle the pattern has a dependence in ϕ of the form $\cos(\phi - \tilde{\phi})$. The phase $\tilde{\phi}$ depends on θ, and hence the spiralling of the simple cold and hot bump induced by the $\cos\phi$ dependence. The functions $f_1(\theta)$ and $\tilde{\phi}(\theta)$ are rather complicated functions which have to be evaluated numerically, and depend on various details of the particular Bianchi model within the type we have chosen. It is curious to note, however, that only the power spectrum C_ℓ and the phases ϕ are sensitive to these details. All the spirals imprinted by Bianchi VII$_h$ models have moduli of the form

$$\rho^\ell_m = \delta_{m1} f_2(\ell, x) \tag{35}$$

Therefore their shape spectra will always be

$$\begin{aligned} B^\ell_m &= 1 \qquad \text{for} \quad 2 \leq m \leq \ell \\ B^\ell_m &= 0 \qquad \text{for} \quad m = 1 \end{aligned} \tag{36}$$

The background patterns in Bianchi VII$_h$ models are all localized, shapeful, and connected structures. Depending on the model they will however have different positions, power spectra, and connectivity. Nevertheless, their shape spectra is always the same exact shape, of form (36), without any cosmic variance error bars. Confusion with a Gaussian is zero. Confusion with the shape of a perfect texture hot spot is zero as well. These have a non-Gaussian spectrum of the form

$$B^\ell_m = 1 \qquad \text{for} \quad 1 \leq m \leq \ell \tag{37}$$

Although the shape spectrum is the same up to the last B^ℓ_m, the confusion between the two theories is zero.

VII. AN ANISOTROPIC GRISCHUK-ZEL'DOVICH EFFECT

We shall now consider an example of a flat homogeneous and isotropic universe with topological identification along one axis. This example is simpler than most of those considered in the literature but illustrates one of the key features of such models: the breaking of statistical isotropy in the fluctuations. Let us consider a universe with a topological identification along the z axis. All functions defined on such a space satisfy:

$$\Phi(x,y,z) = \Phi(x,y,z+L) \tag{38}$$

By considering a flat universe we can restrict ourselves to calculating the Sachs-Wolfe effect. The temperature anisotropy from the surface of last scattering will be given be:

$$\frac{\Delta T}{T}(\mathbf{n}) = -\frac{1}{2}\frac{H_0^2}{c^2}\int d^2k \sum_{j=-\infty}^{+\infty} \delta_{\mathbf{k}j} \frac{e^{i\Delta\mathbf{n}\cdot\mathbf{q}}}{q^2} \tag{39}$$

where $\Delta = \eta_0 - \eta_{ls}$ is radius of the surface of last scatter and $q = (k\cos\phi, k\sin\phi, 2\pi\frac{j}{L})$. We can expand the exponential in spherical harmonics to get

The $a_{\ell m}$s are given by

$$a_{\ell m} = -\frac{i^{-\ell}}{2}\frac{H_0^2}{c^2}\int d^2k \sum_j \delta_{\mathbf{k}j}\frac{j_\ell(\Delta q)}{q^2}Y^*_{\ell m}(\hat{q}) \tag{40}$$

We now assume statistical homogeneity and isotropy of δ and a scale invariant power spectrum

$$\langle \delta^*_{\mathbf{k}j}\delta_{\mathbf{k}'j'}\rangle = \delta^2(\mathbf{k}-\mathbf{k}')\delta_{jj'}q^{-1} \quad q = \sqrt{k^2 + (\frac{2\pi j}{L})^2} \tag{41}$$

This leads to the covariance matrix for the $a_{\ell m}$s:

$$\langle a^*_{\ell m} a_{\ell' m'}\rangle = \frac{i^{\ell'-\ell}}{4}\frac{H_0^4}{c^4}\int d^2k \sum_j \frac{j_\ell(q\Delta)j_{\ell'}(q\Delta)}{q^3}Y^*_{\ell m}(\hat{q})Y_{\ell' m'}(\hat{q}) \tag{42}$$

Expressing $Y_{lm}(\mathbf{n}) = \tilde{P}_{lm}(\cos\theta)e^{im\phi}$ and performing the azimuthal integral, one immediately finds that the covariance matrix is diagonal in m, so one has

$$\langle a^*_{\ell m} a_{\ell' m'}\rangle = \frac{i^{\ell'-\ell}}{4}\frac{H_0^4}{c^4}\delta_{mm'}\int kdk \sum_j \frac{j_\ell(q\Delta)j_{\ell'}(q\Delta)}{q^3}\tilde{P}_{\ell m}(\hat{q})\tilde{P}_{\ell' m'}(\hat{q}) \tag{43}$$

It is convenient to define $\chi = \Delta k$ and $\mu_j = 2\pi\alpha j$ where $\alpha = \Delta/L$, i.e. the ration of our horizon to the topological identification scale. If we now define $y = \mu_j/\sqrt{\mu_j^2 + \chi^2}$ the expression simplifies to

$$\langle a^*_{\ell m} a_{\ell' m'}\rangle = \frac{i^{\ell'-\ell}}{4}\frac{H_0^4\Delta}{c^4}\delta_{mm'}\int_0^1 dy \sum_j \frac{j_\ell(\frac{\mu_j}{y})j_{\ell'}(\frac{\mu_j}{y})}{\mu_j}\tilde{P}_{\ell m}(y)\tilde{P}_{\ell' m'}(y) \tag{44}$$

In the limit where the identification scale goes to infinity we get the standard result

$$\lim_{\alpha\to 0}\langle a^*_{\ell m} a_{\ell' m'}\rangle \propto \int_0^1 dy j_\ell^2(\frac{1}{y})\delta_{\ell\ell'}\delta_{mm'} \propto \frac{1}{\ell^2}\delta_{\ell\ell'}\delta_{mm'} \tag{45}$$

i.e. a scale invariant, diagonal covariance matrix. In the case of finite α this is not the case. Consider the quadrupole. The ring spectra has two components, B_1 and B_2 with a probability distribution function

$$F(r, B_1, B_2) = \frac{\exp(-r^2/2\sigma_2^2)r^4}{(\pi/2)^{1/2}\sigma_2^3 3!!} \times \exp\{-(r^2/2\sigma_2^2)[c_2 B_2^2(1 + \frac{c_1}{c_2}B_1^{2/3})]\} \tag{46}$$

where we have defined

$$\begin{aligned} \sigma_2^2 &= (\langle|a_{20}|^2\rangle\langle|a_{21}|^2\rangle\langle|a_{22}|^2\rangle)^{1/3} \\ c_i &= \frac{\sigma_2^2}{\langle|a_{2i-1}|^2\rangle} - \frac{\sigma_2^2}{\langle|a_{2i}|^2\rangle} \end{aligned} \tag{47}$$

By exploring the dependence of c_1 and c_2 on α we can see how the probability distribution function of the Bs change with topology scale; to a very good approximation we find

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{\sqrt{2}}{2} \\ c_2 &= \frac{2\alpha}{1+\alpha} \end{aligned} \tag{48}$$

We can see the signature for non-Gaussianity arising here. For a non zero α there are correlations between the three statistical quantities, in particular $\langle B_1 B_2\rangle \propto \alpha/(1+\alpha)$. The probability distribution function for B_1 and B_2 (defined on $[0,1]\times[0,1]$) becomes peaked at 0. We have focused on the quadrupole where the effect is easy to see. The method is systematic however, and one can construct the probability distribution function of the high order ring spectra in the same way.

This a curious application of the idea put forward in [21], an anisotropic Grischuk-Zel'Dovich effect. By looking at the shape of the low ℓ multipole moments we can constrain the degree of statistical anisotropy outside the current horizon. Note that, already for $\alpha < 1$ there are deformations in the covariance matrix which may be statistically significant with current data. This will be pursued in a future publication [22].

VIII. CONCLUSIONS

In this paper we have presented a new technique for quantifying non-Gaussianity on large scales. It is the extension of the non-Gaussian spectra developed in [19] to the surface of the two sphere. As we have shown in Section II the construction is slightly different to take into account the particularities of the spherical harmonic basis. However the qualititative interpretation of the different levels of non-Gaussianity follows through, exactly as in [19]. One can identify the information contained in the ring, interring and phase spectra with shape, connectivity and localization.

An interesting and untapped application is to universes with statistically anisotropic fluctuations. Developing the idea put forward in [10] we explain how statistical anisotropy and non-Gaussianity are intimately related. From this one can infer some novel properties of the covariance matrix of fluctuations in statistically anisotropic spacetimes. In particular, features which appear in non-gaussian theories of structure formation, like textures [11,12] will appear here: a surplus of cosmic variance and cosmic *co*-variance of the power spectra.

There are a number of situations where these results are applicable. One is in the case of anisotropic universes, i.e. universes which aren't Friedman-Robertson-Walker. There are a number of known examples [1,2]. Without actually doing perturbation theory on them we argue for a natural prescription for adding fluctuations in the CMB to such models. It consists of finding the reduced symmetry group of the temperature patterns and constructing anisotropic Gaussian random fields with such properties (in which the covariance matrix satisfies those symmetries, and not more). In fact it can be shown that these symmetries can be deduced from the geodesic structure of the space time [23]. This can be a first step in extending the prescription used in [6] for constraining general anisotropic models with the COBE 4 year data. A brief analysis is made of the relevant Bianchi models for which we present the non-Gaussian spectra.

Another, different application is the case of homogeneous isotropic models where an anisotropic topological identification has been imposed. As an example we identify one direction in space. One finds that statistical isotropy is broken. This can be easily from the following: if we look along the axis of identification, and the identification scale is smaller than our horizon, one will find strong correlations between patches of the microwave sky which are reflected about the uncompactified plane [7]. By looking at the structure of the covariance matrix one can see that this anisotropy will manifest itself by inducing not only non-Gaussian ring spectra but also inter-ring spectra. This non-Gaussian manifestation may persist if we consider the identification scale to be large than our horizon. We name this effect the *anisotropic* Grischuk-Zel'Dovich effect.

Although we now have a high quality measurement of anisotropies on large angular scales we are confronted with the hardships of the real world. Galactic contamination leads one to consider an anisotropic rendition of the sky and considerably complicates the analysis of the COBE four year data. It is well known that one of the consequences is that the quadrupole measurement should be viewed with scepticism. Unfortunately it is the quadrupole which could supply us with a probe of primordial CMB on the largest angular scales. There may be ways around these shortcomings. One can try and reformulate our non-Gaussian spectra on the largest angular scales using the techniques put forward in [24]. This would involve a proper likelihood analysis and to make the problem tractable one would need to find an adequate parametrization of the non-Gaussian spectra. The fact that we have devised a consistent method for characterizing non-Gaussianity on all scales may allow us to use the cumulative information of all $\ell > 2$ to infer the behaviour on large scales; all modes will be affected to some extent by large scale anisotropy. Finally it would be interesting to analyse in more detail the observability of the anisotropic Grischuk-Zel'Dovich effect, taking into consideration issues of cosmic variance.

ACKNOWLEDGEMENTS

We would like to thank T Kibble for pointing out and correcting a serious mistake at the heart of this paper. J.M. thanks MRAO-Cambridge for use of computer facilities while this paper was being prepared. P.G.F gratefully acknowledges insightful comments from C. Balland, J. Levin, A. Jaffe and J. Silk. P.G.F. was supported by the Center for Particle Astrophysics, a NSF Science and Technology Center at UC Berkeley, under Cooperative Agreement No. AST 9120005. J.M. was supported by a Royal Society University Research Fellowship.

APPENDIX I - GAUSSIAN FLUCTUATIONS WITH A SYMMETRY

In this Appendix we detail the group theory argument sketched in Section III. This argument is not necessary in the $SO(2)$ case targeted in this paper, where the covariance matrix may be easily derived directly. However, it opens

doors to more general anisotropic symmetry groups. It also shows how the general form of the covariance matrix depends not on the symmetry group, but only on its irreducible representations (Irreps).

Fluctuations (Gaussian or not) in any Universe must be subject to the symmetries of the underlying cosmological model. However, due to the random nature of the fluctuations, they must satisfy these symmetries only statistically. By this we mean that the statistical ensemble of fluctuations, and not each realization, should be subject to the symmetries. If a symmetry transformation is applied to each member of the ensemble, then each member may change, but the ensemble should remain the same. For instance, for the $SO(3)$ symmetry group only realizations containing only a monopole $\ell = 0$ are left unchanged by rotations. Nevertheless much more general fluctuations respect statistical isotropy. Similarly, only the $m = 0$ modes are cylindrically symmetric, but more general fluctuations are $SO(2)$ statistically invariant[5].

Gaussian fluctuations are fully specified by their covariance matrix

$$C^{\ell_1 \ell_2}_{m_1 m_2} = \langle a^{\ell_1}_{m_1} a^{\ell_2 \star}_{m_2} \rangle \tag{49}$$

which may be seen as a bilinear form on the $\{a^\ell_m\}$ space. Hence the statistical symmetries of a Gaussian theory are equivalent to the requirement that the covariance matrix is left unchanged by any symmetry transformation. Let G be the symmetry group of the underlying cosmological model as projected on the sky. Let's first suppose that G breaks the $\{a^\ell_m\}$ into a set on non-equivalent Irreps. Then, let's find a new basis a^L_M adapted to G, where L now labels the Irrep the basis element belongs to, and M the actual element. Then G is represented by a set of matrices $G^L_{MM'}$ acting on a^L_M as

$$\tilde{a}^L_M = G a^L_M = G^L_{MM'} a^L_{M'} \tag{50}$$

The covariance matrix for the a^L_M

$$C^{L_1 L_2}_{M_1 M_2} = \langle a^{L_1}_{M_1} a^{L_2 \star}_{M_2} \rangle \tag{51}$$

must remain unchanged by the transformation (50), so that

$$\tilde{C}^{L_1 L_2}_{M_1 M_2} = \langle \tilde{a}^{L_1}_{M_1} \tilde{a}^{L_2 \star}_{M_2} \rangle = G^{L_1}_{M_1 M_1'} G^{L_2 \star}_{M_2 M_2'} C^{L_1 L_2}_{M_1' M_2'} = C^{L_1 L_2}_{M_1 M_2} \tag{52}$$

which for unitary representations amounts to the commutation relation

$$G^{L_1} C^{L_1 L_2} = C^{L_1 L_2} G^{L_2} \tag{53}$$

Let us now recall Schur's Lemmas [20].

Schur's Lemma 1 *Let Γ and Γ' be two Irreps of a group G with dimensions d and d', and let there be a $d \times d'$ matrix A such that*

$$\Gamma(g) A = A \Gamma'(g) \tag{54}$$

for all group elements g. Then either $A = 0$ or $d = d'$ and $\det A \neq 0$.

It follows that if $A \neq 0$ then Γ and Γ' are equivalent.

Schur's Lemma 2 *If Γ is a d-dimensional Irrep of a group G and B is a $d \times d$ matrix such that*

$$\Gamma(g) B = B \Gamma(g) \tag{55}$$

for all group elements g, then $B = \lambda 1$.

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[5]Some texture models exhibit approximate $SO(2)$ symmetry at low ℓ in *each* realization. Then an axes system exists in which the $m = 0$ mode has much more power than any other. This is of course a very non-Gaussian effect which cannot be simply reproduced by anisotropic Gaussian fluctuations.

Combining the two Lemmas we can then find that the covariance matrix must take the form

$$C^{L_1L_2}_{M_1M_2} = \langle \tilde{a}^{L_1}_{M_1}\tilde{a}^{L_2\star}_{M_2}\rangle = \delta^{L_1L_2}\delta_{M_1M_2}C_{L_1} \tag{56}$$

An interesting result is that if G breaks the $\{a^\ell_m\}$ space into non-equivalent Irreps spanned by $\{a^L_M\}$, then these must be independent random variables with a variance which can only depend on the Irrep they belong to. This argument applies for instance if the symmetry group is $SO(3)$, in which case $L=\ell$ and $M=m$.

The argument just present breaks down however, if some of the Irreps defined by G are equivalent. Let the $\{a^\ell_m\}$ space now be spanned by an adapted basis $\{a^{LD}_M\}$, where L labels each class of equivalent representations, D labels the actual Irrep within this class, and M labels the elements within each Irrep. Then, from the second Schur's Lemma we know that within the same Irrep we still have:

$$\langle a^{LD}_{M_1}a^{LD\star}_{M_2}\rangle = \delta_{M_1M_2}C^{LD} \tag{57}$$

but for different but equivalent Irreps ($D_1 \neq D_2$) we now have

$$\langle a^{LD_1}_{M_1}a^{LD_2\star}_{M_2}\rangle = C^{LD_1D_2}_{M_1M_2} \tag{58}$$

with det$C \neq 0$, whereas for non-equivalent Irreps ($L_1 \neq L_2$) we still have

$$\langle a^{L_1D_1}_{M_1}a^{L_2D_2\star}_{M_2}\rangle = 0 \tag{59}$$

Therefore, although the a^{LD}_M are independent within each Irrep and among different non-equivalent Irreps, correlations may exist between different but equivalent Irreps. Of course one may always rotate the a^{LD}_M within each class of equivalent Irreps so as to diagonalize the covariance matrix. However such a rotation is model dependent, and cannot be determined from the symmetries. This situation happens for instance in the case of $SO(2)$, with $L=m$ and $D=\ell$ (no M index, since the Irreps are one dimensional). Each m provides a class of equivalent representations, with the same m but different ℓ. Then the covariance matrix takes the general form

$$\langle a^{\ell_1}_{m_1}a^{\ell_2\star}_{m_2}\rangle = \delta_{m_1m_2}C^{\ell_1\ell_2}_{m_1} \tag{60}$$

and as we see although correlations among different m are not allowed, now we may have correlations between different ℓ, for the same m. We could rotate the a^ℓ_m in ℓ for each fixed m so as to diagonalize the covariance matrix, but such procedure naturally would depend on the covariance matrix one starts from, and would therefore be model dependent.

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